JEE Main 2023 (Memory based)

31 January 2023 - Shift 1

Answer & Solutions

Mathematics

1. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2+3\sin x}{\sin x(1+\cos x)} dx$ is equal to:

A. $\ln(\sqrt{3}+2) - \frac{\ln 3}{2} + 6\sqrt{3} - \frac{28}{3}$
B. $\ln(\sqrt{3}+2) - \frac{\ln 3}{2}$
C. $\ln(\sqrt{3}+2) - \frac{\ln 3}{2} - \frac{28}{3}$
D. $6\sqrt{3} - \frac{28}{3}$

Answer (A)
Solution:

$$I = \int \frac{2}{\sin x(1+\cos x)} dx + \int \frac{3}{(1+\cos x)} dx$$

$$= \int \frac{2\sin x}{\sin^2 x(1+\cos x)} dx + \int \frac{3}{2\cos^2\frac{x}{2}} dx$$

Let I_1 and I_2 be the first and second integral respectively.

Let $\cos x = t$

$$I_1 = \int \frac{-2dt}{(1-t^2)(1+t)}$$

$$I_1 = -2\left(\frac{\ln(t+1)}{4} + \frac{1}{2t+2} - \frac{\ln|t-1|}{4}\right) + C$$

$$I_1 = -2\left(\frac{\ln(t+1)}{4} + \frac{1}{2t+2} - \frac{\ln|t-1|}{4}\right) + C$$

$$I_1 = -2\left(\frac{\ln(\cos x+1)}{4} + \frac{1}{2\cos x+2} - \frac{\ln|\cos x-1|}{4}\right) + C$$

$$I_2 = \frac{3}{2}\left(2\tan\frac{x}{2}\right) + C$$

So, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2+3\sin x}{\sin x(1+\cos x)} dx = \ln(\sqrt{3}+2) - \frac{\ln 3}{2} + 6\sqrt{3} - \frac{28}{3}$

2. The product and sum of first four terms of G.P. is 1296 and 126 respectively, then sum of the possible values of common difference is:

A. 14
B. $\frac{10}{3}$
C. $\frac{7}{2}$
D. 3

Answer (D)

Solution:

$$\frac{a}{r^3}.\frac{a}{r}.ar.ar^3 = 1296$$

$$\Rightarrow a = 6$$

$$\frac{a}{r^3} + \frac{a}{r} + ar + ar^3 = 126$$

$$\Rightarrow \frac{1}{r^3} + \frac{1}{r} + r + r^3 = 21$$

$\Rightarrow \left(r+\frac{1}{r}\right)\left(\left(r+\frac{1}{r}\right)^2-3\right)+\left(r+\frac{1}{r}\right)=21$

Let $r+\frac{1}{r}=t$

$\Rightarrow t^3-3t+t=21$

$\Rightarrow t^3-2t-21=0$

$\Rightarrow t=3$

$\Rightarrow r+\frac{1}{r}=3$

$\Rightarrow r^2-3r+1=0$

$\Rightarrow r_1+r_2=3$

Sum of possible values of r is 3.

3. If $B=\ln(1-a)$ and $P(a)=\left(a+\frac{a^2}{2}+\frac{a^3}{3}+\cdots+\frac{a^{50}}{50}\right)$, then $\int_0^a \frac{t^{50}}{1-t}dt$ equals:

A. $-\left(B+P(a)\right)$
B. $-B+P(a)$
C. $B-P(a)$
D. $B+P(a)$

Answer (A)

Solution:

$\int_0^a \frac{t^{50}}{1-t}dt=\int_0^a \frac{t^{50}-1+1}{1-t}dt$

$\Rightarrow \int_0^a \left(\frac{t^{50}-1}{1-t}+\frac{1}{1-t}\right)dt$

Since $(1+t+t^2+\cdots+t^{49})$ constitute as a G.P. with sum $=\frac{t^{50}-1}{t-1}$

$\Rightarrow \int_0^a \left(-(1+t+t^2+\cdots+t^{49})+\frac{1}{1-t}\right)dt$

$\Rightarrow [-\ln(1-t)]_0^a-\left[\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^{50}}{50}\right)\right]_0^a$

$\Rightarrow -\ln(1-a)-\left(a+\frac{a^2}{2}+\frac{a^3}{3}+\cdots+\frac{a^{50}}{50}\right)\Rightarrow -\left(B+P(a)\right)$

4. $\sin^{-1}\left(\frac{a}{17}\right)+\cos^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{77}{36}\right)=0$, then the value of $\sin^{-1}(\sin a)+\cos^{-1}(\cos a)$ is :

A. 0
B. $16-2\pi$
C. π
D. 5

Answer (C)

Solution:

$\sin^{-1}\left(\frac{a}{17}\right)=-\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{77}{36}\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right)=\beta$ & $\tan^{-1}\left(\frac{77}{36}\right)=\alpha$

$\Rightarrow \sin\left(\sin^{-1}\frac{a}{17}\right)=\sin(\alpha-\beta)$

$\Rightarrow \sin\left(\sin^{-1}\frac{a}{17}\right)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\Rightarrow \frac{a}{17} = \frac{77}{85} \times \frac{4}{5} - \frac{36}{85} \times \frac{3}{5}$

$\Rightarrow a = \frac{200}{25} = 8$

$\Rightarrow \sin^{-1} \sin 8 + \cos^{-1} \cos 8 = 3\pi - 8 + 8 - 2\pi$

$= \pi$

5. If maximum distance of a normal to the ellipse $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$ from $(0,0)$ is 1, then the eccentricity of the ellipse is:

A. $\frac{\sqrt{3}}{4}$
B. $\frac{1}{\sqrt{2}}$
C. $\frac{1}{2}$
D. $\frac{\sqrt{3}}{2}$

Answer (D)

Solution:

Equation of normal is

$(2 \sec \theta)x - (b \ \operatorname{cosec} \theta)y = 4 - b^2$

Perpendicular distance from $(0,0)$ is

$D = \left|\frac{4-b^2}{\sqrt{4 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta}}\right|$

$= \frac{4-b^2}{\sqrt{(4+b^2)+4\tan^2 \theta + b^2 \cot^2 \theta}} \leq \frac{4-b^2}{\sqrt{b^2+4+4b}}$ (using AM $\geq$ GM for $4 \tan^2 \theta + b^2 \cot^2 \theta$)

$= \frac{4-b^2}{(2+b)}$

$= 2 - b$

$D_{max} = 2 - b = 1$

$\Rightarrow b = 1$

$\Rightarrow e = \sqrt{1 - \frac{1}{4}}$

$\Rightarrow e = \frac{\sqrt{3}}{2}$

6. Let the curve C_1 be represented by $|z| = 2$ and C_2 by $\left|z + \frac{\bar{z}}{4}\right| = \frac{15}{4}$, then:

A. C_1 lies inside C_2
B. C_2 lies inside C_1
C. C_1 & C_2 has 2 points of intersections.
D. C_1 & C_2 has 4 points of intersections.

Answer (A)

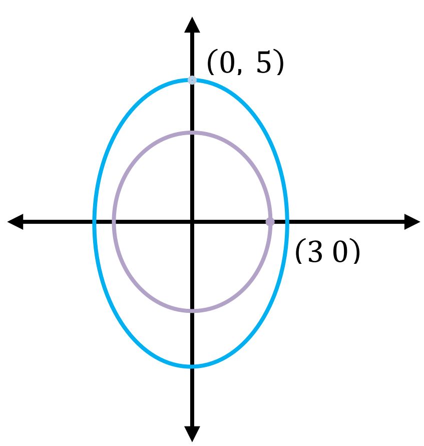
Solution:

Let $z = x + iy$

$C_1 \Rightarrow x^2 + y^2 = 4 \Rightarrow$ circle

$C_2 \Rightarrow \left|x + iy + \frac{x-iy}{4}\right| = \frac{15}{4}$

$\Rightarrow \left(\frac{5x}{4}\right)^2 + \left(\frac{3y}{4}\right)^2 = \frac{225}{16}$



$\Rightarrow \frac{x^2}{9}+\frac{y^2}{25}=1 \Rightarrow$ ellipse
$\Rightarrow C_1$ lies inside C_2

7. Find the number of real solutions of $\sqrt{x^2-4x+3}+\sqrt{x^2-9}=\sqrt{4x^2-14x+6}$ is:

A. 1
B. 2
C. 3
D. 4

Answer (A)

Solution:

Effectively, the network is
$x^2-4x+3 \geq 0$
$\Rightarrow (x-1)(x-3) \geq 0$
$\Rightarrow x \in (-\infty,\ 1] \cup [3,\ \infty) \cdots (1)$
$x^2-9 \geq 0$
$\Rightarrow x \in (-\infty,\ -3] \cup [3,\ \infty) \cdots (2)$
$4x^2-14x+6 \geq 0$
$\Rightarrow (2x-1)(x-3) \geq 0$
$\Rightarrow x \in \left(-\infty, \frac{1}{2}\right] \cup [3,\ \infty) \cdots (3)$
$(1) \cap (2) \cap (3)$, we get
$x \in (-\infty,\ -3] \cup [3,\ \infty)$
Now squaring both sides of given equation.
$(x^2-4x+3)+(x^2-9)+2\sqrt{(x^2-4x+3)(x^2-9)}=4x^2-14x+6$
$\Rightarrow 2\sqrt{(x^2-4x+3)(x-3)(x+3)}=2(x^2-5x+6)$
$\Rightarrow (x^2-4x+3)(x-3)(x+3)=(x-3)^2(x-2)^2$
$x=3$ is one solution
$\Rightarrow (x^2-4x+3)(x+3)=(x^2-4x+4)(x-3)$
$\Rightarrow x^3-4x^2+3x+3x^2-12x+9=x^3-4x^2+4x-3x^2+12x-12$
$\Rightarrow 6x^2-25x+21=0$
$\Rightarrow x=3, \frac{7}{6}$
$x=\frac{7}{6}$ is not in domain. So, only one solution.

8. If $f(x)=\sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}(-4x^3+5x^2+1)^{\frac{3}{2}}\right)\right)$, then $f'(1)$ is :

A. $\frac{3\pi^2}{8}$
B. $\frac{3\pi^2}{4}$
C. $\frac{3\pi^2}{16}$
D. $\frac{\pi^2}{2}$

Answer (C)

Solution:

$$f(x) = \sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}(-4x^3+5x^2+1)^{\frac{3}{2}}\right)\right)$$

$$f'(x) = 3\sin^2\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}(-4x^3+5x^2+1)^{\frac{3}{2}}\right)\right) \times \cos\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}(-4x^3+5x^2+1)^{\frac{3}{2}}\right)\right) \times$$
$$\frac{\pi}{3}\left(-\sin\left(\frac{\pi}{3\sqrt{2}}(-4x^3+5x^2+1)^{\frac{3}{2}}\right)\right) \times \frac{\pi}{3\sqrt{2}} \times \frac{3}{2}(-4x^3+5x^2+1)^{\frac{1}{2}} \times (-12x^2+10x)$$

$$f'(1) = 3\sin^2\left(\frac{\pi}{3}\cos\left(\frac{2\pi}{3}\right)\right) \times \cos\left(\frac{\pi}{3}\cos\left(\frac{2\pi}{3}\right)\right) \times \frac{\pi}{3}\left(-\sin\frac{2\pi}{3}\right) \times \frac{\pi}{2\sqrt{2}}(\sqrt{2})(-2)$$

$$f'(1) = 3\sin^2\left(-\frac{\pi}{6}\right) \times \cos\left(-\frac{\pi}{6}\right) \times \frac{\pi}{3}\left(-\frac{\sqrt{3}}{2}\right) \times (-\pi)$$

$$f'(1) = \frac{3}{4} \times \frac{\sqrt{3}}{2} \times \frac{\pi}{3} \times \frac{\sqrt{3}}{2} \times \pi = \frac{3\pi^2}{16}$$

9. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that $|\vec{a}+\vec{b}+\vec{c}| = |\vec{a}+\vec{b}-\vec{c}|$ and $\vec{b}\cdot\vec{c} = 0$ then:

S-I: $|\vec{a}+\lambda\vec{c}| \geq 0$ for all $\lambda \in \mathbb{R}$

S-II: $\vec{a}$ is always parallel to $\vec{c}$

A. S-I is True, S-II is False.

B. S-I is True, S-II is True.

C. S-I is False, S-II is True.

D. S-I is False, S-II is False.

Answer (A)

Solution:

$|\vec{a}+\vec{b}+\vec{c}| = |\vec{a}+\vec{b}-\vec{c}|$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}+\vec{b}-\vec{c}|^2$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{c}-\vec{c}\cdot\vec{a}$

$\Rightarrow \vec{c}\cdot\vec{a} = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{c}$

$\Rightarrow$ S-II is False.

$\Rightarrow |\vec{a}+\lambda\vec{c}| \geq 0$ (However, it is always true)

$\Rightarrow$ S-I is True.

10. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then find sum of diagonal elements of $(A-I)^{11}$.

A. 4096

B. 4097

C. 2048

D. 2049

Answer (D)

Solution:

$A - I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$(A-I)^{11}=\begin{bmatrix}1^{11} & 0 & 0\\ 0 & 2^{11} & 0\\ 0 & 0 & 0^{11}\end{bmatrix}$$

$trace\ (A-I)^{11}=2^{11}+1^{11}+0$

$trace\ (A-I)^{11}=2049$

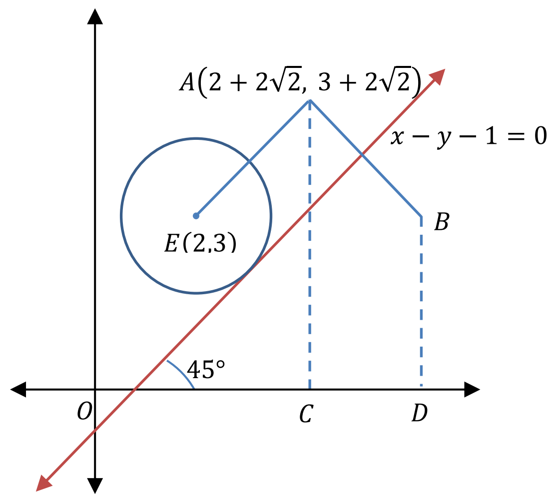
11. Circle $x^2+y^2-4x-6y+11=0$ is rolled up by 4 units along a tangent to it at the point $(3,\ 2)$. Let this be circle C_1. C_2 is the mirror image of circle C_1 about the tangent. A and B are centres of circles C_1 and C_2. C and D are the feet of perpendiculars from A and B respectively upon X-axis. The area of the trapezium $ABCD$ equals to:

A. $4(1+\sqrt{2})$

B. $2(1+\sqrt{2})$

C. $3(1+\sqrt{2})$

D. $(1+\sqrt{2})$



Answer (A)

Solution:

Given circle is $x^2+y^2-4x-6y+11=0$, Centre $E(2,3)$

Tangent at $(3,2)$ is $x-y-1=0$

After rolling up by 4 units, centre of C_1 is A

Where $A\equiv\left(2+4\times\frac{1}{\sqrt{2}},3+4\times\frac{1}{\sqrt{2}}\right)\equiv\left(2+2\sqrt{2},3+2\sqrt{2}\right)$

B is image of A about $x-y-1=0$

$\frac{x-(2+2\sqrt{2})}{1}=\frac{y-(3+2\sqrt{2})}{-1}=-2\times\left(\frac{-2}{2}\right)=2$

$B\equiv\left(4+2\sqrt{2},1+2\sqrt{2}\right)$

Area of $ABCD=\frac{1}{2}\times\left(4+4\sqrt{2}\right)\times\left(\left(4+2\sqrt{2}\right)-\left(2+2\sqrt{2}\right)\right)$

Area of $ABCD=4\left(1+\sqrt{2}\right)$

12. Let the solution R, $(a,\ b)\ R(c,\ d)$ be such that $ab(d-c)=cd(a-b)$ then R is:

A. Reflexive only

B. Symmetric only

C. Transitive but not symmetric

D. Reflexive and symmetric but not transitive

Answer (B)

Solution:

Checking for Reflexive

$\therefore (a,b)\ R\ (a,b)$

$\Rightarrow ab(b-a)=ab(a-b)$

$\Rightarrow b-a=a-b$ $\therefore$ Not reflexive

Checking for $(a,b)\ R\ (c,d)$ then $(c,d)\ R\ (a,b)$

$\Rightarrow cd(b-a)=ab(c-d)$

$\Rightarrow ab(d-c)=cd(a-b)$

$\therefore R$ is symmetric.

$(a, b)\ R\ (c, d) \equiv ab(d-c) = cd(a-b)$

$\Rightarrow \frac{ab}{a-b} = \frac{cd}{d-c} \quad \cdots (1)$

$(c, d)\ R\ (e, f) \equiv \frac{cd}{c-d} = \frac{ef}{f-e} \quad \cdots (2)$

For relation to be transitive, we need to check whether $(a, b)\ R\ (e, f)$ or not.

$i.e. \frac{ab}{a-b} = \frac{ef}{f-e}$

But, by (1) and (2) we get,

$\frac{ab}{a-b} = -\frac{ef}{f-e}$

$\therefore R$ is not transitive.

13. Find the number of 5-digit numbers formed using the digits $0, 3, 4, 7, 9$ when repetition of digits is allowed:

Answer (2500)

Solution:

↓	↓	↓	↓	↓
3	0	0	0	0
4	3	3	3	3
7	4	4	4	4
9	7	7	7	7
	9	9	9	9

Total number of 5-digit numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$

14. Reminder when 5^{99} is divided by 11 is ___________.

Answer (9)

Solution:

We have,

$5^{99} = (5^5)^{19} \cdot 5^4$

$= (3125)^{19} \cdot 5^4$

$= (11\lambda + 1)^{19} \cdot 5^4$

$= (11k + 1) \cdot 5^4$

$= 11k_1 + 5^4$

When 5^4 is divisible by 11 we get remainder $= 9$

15. If $f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}$ then value of $12f(8)$ equals __________.

Answer (17)

Solution:

$f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}$

Differentiating on both sides,

$f'(x) + \frac{f(x)}{x} = \frac{1}{2\sqrt{x+1}}$

since $y = f(x)$ we get $\frac{dy}{dx} = f'(x)$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{1}{2\sqrt{x+1}}$

Integrating factor(I.F.) $= e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$\Rightarrow xy = \frac{1}{2}\int \frac{x}{\sqrt{x+1}}$
$\Rightarrow xy = \frac{1}{2}\int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}$
$\Rightarrow xy = \frac{1}{2}\left(\frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1}\right) + c \quad \cdots (1)$
If we put $x = 3$ in $f(x) + \int_3^x \frac{f(t)}{t}dt = \sqrt{x+1}$ we get,
$\Rightarrow f(3) + \int_3^3 \frac{f(t)}{t}dt = \sqrt{4}$
$\Rightarrow f(3) = 2$
By substituting $f(3) = 2$ in eq.(1)
$\Rightarrow 3 \times 2 = \frac{1}{2}\left(\frac{2}{3}(4)^{\frac{3}{2}} - 2\sqrt{4}\right) + c$
$\Rightarrow c = \frac{16}{3}$
$\therefore 8f(8) = \frac{1}{2}\left(\frac{2}{3}(9)^{\frac{3}{2}} - 2\sqrt{9}\right) + \frac{16}{3}$
$\Rightarrow 8f(8) = \frac{27}{3} - 3 + \frac{16}{3}$
$\Rightarrow 8f(8) = \frac{34}{3}$
$\Rightarrow 12f(8) = 17$

16. $y = f(x)$ is a parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $y = -\frac{1}{2}$. Given that $\tan^{-1}\sqrt{f(x)} + \sin^{-1}\sqrt{f(x)+1} = \frac{\pi}{2}$. Then number of real solutions for x is _____.

Answer (2)

Solution:

$SP = SQ$
$\left(x + \frac{1}{2}\right)^2 + y^2 = \left(y + \frac{1}{2}\right)^2$
$x^2 + \frac{1}{4} + x + y^2 = y^2 + \frac{1}{4} + y$
Equation of parabola: $y = x^2 + x$
$\Rightarrow f(x) = x^2 + x$
$\Rightarrow \tan^{-1}\sqrt{x^2+x} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$
$\Rightarrow \tan^{-1}\sqrt{x^2+x} = \cos^{-1}\sqrt{x^2+x+1}$
$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{x^2+x}}\right) = \cos^{-1}\sqrt{x^2+x+1}$
$\Rightarrow \left(\frac{1}{\sqrt{x^2+x}}\right) = \sqrt{x^2+x+1}$
$\Rightarrow x^2 + x + 1 = 1$
$\Rightarrow x^2 + x = 0$
$\Rightarrow x(x+1) = 0$
$\Rightarrow x = 0$ or -1
$\therefore$ Number of real solutions for x are 2.

17. The direction ratio's of two lines which are parallel are given by $< 2, 1, -1 >$ and $< \alpha+\beta, 1+\beta, 2 >$. Then the value of $|2\alpha + 3\beta|$ is ___________.

Answer (11)

Solution:

Since, the lines are parallel.
$\therefore \frac{\alpha+\beta}{2} = \frac{1+\beta}{2} = \frac{2}{-1}$
$\Rightarrow \alpha + \beta = -4$ and $1 + \beta = -2$
$\Rightarrow \beta = -3$
$\Rightarrow \alpha = -1$
So, $|2\alpha + 3\beta| = |2(-1) + 3(-3)| = |-11| = 11$

18. Given $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = \sqrt{6}$, $|\vec{a} \times \vec{b}| = \sqrt{48}$. Then the value of $(\vec{a} \cdot \vec{b})^2$ is ______.

Answer (36)

Solution:

$$(\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \times \vec{b}|^2$$
$$= 14 \times 6 - 48$$
$$= 36$$